5.2 Convergence in Measure.

Definition: $f_n \to f$ in meas if for each $\epsilon > 0$

$$\lim_{n\to\infty} m(x \in E \mid |f_n - f| > \epsilon) = 0$$

The set of points where $|f_n - f| > \epsilon$ is very small in measure.

If $f_n \to f$ uniformly then of course

$f_n \to f$ in measure. since

$m(\{x \mid |f_n - f| < \epsilon\}) = 1$ for large enough n.

Prop: If $m(E) < \infty$ and $f_n \to f$ a.s then

$f_n \to f$ in meas.

Pf: Only requires continuity of measure.

$$E_m = \bigcap_{n \geq m} \{x \mid |f_n - f| < \eta\}$$

is an increasing sequence. and $\bigcup^{\infty} E_m$ corresponds to $|f_n - f|$ being smaller than η eventually.

Then $E_m^c = \bigcup_{n \geq m} \{x \mid f_n - f| > \eta\}$

is decreasing. $E_m^c \subset E$ and hence $m(E_m^c) < \infty$

Then by decreasing continuity of measure,

$$\lim_{m\to\infty} m(E_m^c) = m\left(\bigcap E_m^c\right) = m(E) - m\left(\bigcup E_m\right)$$
$$= 0$$

So we have

$$\lim_{n\to\infty} m(|f_n - f| > \eta) \leq \lim_{n\to\infty} m(E_m^c) = 0.$$

Another way to see this is through Egoroff's Theorem.

Since $m(E) < \infty$ $\exists$ a good set on which $f_n \to f$ uniformly

st $m(E \setminus E_0) < \epsilon$. So pick n st $|f_n - f| < \epsilon$ on E_0

Then $m(|f_n - f| > \eta) = m(E \setminus E_0 \cap \{|f_n - f| > \eta\})$

$+ m(E_0 \cap \{|f_n - f| > \eta\})$

$< \epsilon$

Since ϵ is arbitrary we have completed the proof.

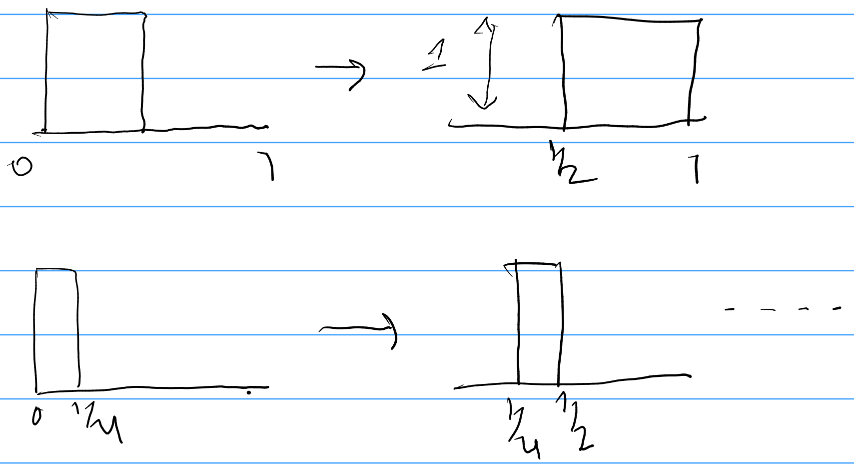
Notice that Egoroff required $m(E) < \infty$

Example: If $m(E) = +\infty$ then above is not true.

Let $f_n = 1_{[n,\infty)}$ $f_n \to 0$ a.s but

$m(|f_n| > \eta) = +\infty$ $\forall n$ and $\eta < 1$.

Example: Traveling interval:



The intervals travel across $[0,1]$ and decrease in size (they halve) each time they wrap around.

So eventually $m(|f_n - 0| > \varepsilon) \to 0$.
But at no point x do we have $f_n(x) \Rightarrow f(x)$ since they will constantly be swept by one of the intervals and hence will be 1 infinitely often.

Theorem (Riesz) If $f_n \to f$ in meas. Then there is a subsequence st $f_{n_j} \to f$ a.s.

Pf: General strategy to prove a.s convergence:

Borel Cantelli Lemma.

Choose n_k st $m\left(|f_{n_k} - f| > \frac{1}{k}\right) < \frac{1}{2^k}$

Let $E_k = \left\{|f_{n_k} - f| > \frac{1}{k}\right\}$

By the BC lemma, we get

$$m(E_k \text{ i.o.}) = m\left(\bigcap_n \bigcup_{k \geq n} E_k\right) = 0$$

$\Rightarrow$ $|f_{n_k} - f| < \frac{1}{k}$ eventually, a.e.

☆

If you have not covered UI and/or tightness then at least state:

Prop: If $\int |f_n| \to 0$ then $f_n \to 0$ in meas.

(L^1 convergence implies convergence in meas)

Pf: $P(|f_n| > \epsilon) \leq \frac{1}{\epsilon} \int |f_n| \to 0$

☆ Can you find an example where $\int |f_n| \to 0$ but we do not have a.s. convergence.

☆ Can you have $f_n \to 0$ in meas. but $\int |f_n| \not\to 0$ ☆ HW